and velocity are x_o and $\dot{x}_o$, respectively. For this problem, Hamilton's principle is

$$\delta\int_{t_o}^{t_1} (m\dot{x}^2+2Fx)\,dt=0 \tag{4}$$

This is probably the most elementary of all motion problems in which an acceleration is involved. Yet, the functional relation sought, $x=x_o+\dot{x}_o t + (F/2m)t^2$, is not directly available from Hamilton's principle. This exact solution is immediately obtainable from Hamilton's law, Eq. (1) of this paper, directly, without any reference to differential equations.[5-7]

References

[1] Smith, D. R. and Smith, C. V., Jr., "When is Hamilton's Principle an Extremum Principle?" *AIAA Journal,* Vol. 12, Nov. 1974, pp. 1573-76.

[2] Hamilton, W. R., "On a General Method in Dynamics," *Philosophical Transactions of the Royal Society of London,* Vol. 124, 1934, pp. 247-308.

[3] Hamilton, W. R., "Second Essay on a General Method in Dynamics," *Philosophical Transactions of the Royal Society of London,* Vol. 125, 1835, pp. 95-144.

[4] Yourgrau, W. and Mandelstam, S., *Variational Principles in Dynamics and Quantum Theory,* W. B. Saunders Co., Philadelphia, 1968, 3rd. ed.

[5] Bailey, C. D., "A New Look at Hamilton's Principle," *Foundations of Physics* Vol. 5, No. 3, Sept. 1975, pp. 433-451.

[6] Bailey, C. D., "Application of Hamilton's Law of Varying Action," *AIAA Journal*, Vol. 13, to be published.

[7] Bailey, C. D., "The Method of Ritz Applied to the Equation of Hamilton," *Computer Methods in Applied Mechanics and Engineering*, Vol. 7, No. 2, 1976, to be published.

[8] Bailey, C. D., "Exact and Direct Analytical Solutions to Vibrating Systems with Discontinuities," *Journal of Sound and Vibration*, Vol. 44, No. 1, Jan., 1976, to be published.

[9] Bailey, C. D., "Hamilton, Ritz and Elastodynamics," ASME *Journal of Applied Mechanics*, to be published.

Comment on "When is Hamilton's Principle an Extremum Principle?"

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IN Ref. 1, D.R. Smith and C.V. Smith Jr. consider the question of when is Hamilton's principle an extremum principle. In developing their answer, the authors begin with the assumption that Hamilton's principle is a variational principle. It seems appropriate, therefore, to remark that Hamilton's principle cannot always be interpreted as a variational principle.

Hamilton's principle can be a variational principle if the system is holonomic. In other works, the equivalence

$$\int_{t_0}^{t_1} (\delta T-\delta V)\,dt=\delta\int_{t_0}^{t_1} (T-V)\,dt \tag{1}$$

cannot be established unless the system is holonomic. Furthermore, when Hamilton's principle is a variational principle, it states only that the actual motion renders the integral on the right-hand side of Eq. (1) stationary. These facts are discussed at length in Refs. 2 and 3.

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References

[1] Smith, D.R. and Smith Jr., C.V., "When is Hamilton's Principle an Extremum Principle?" *AIAA Journal*, Vol. 12, Nov. 1974, pp. 1573-1576.

[2] Lanczos, C., *The Variational Principles of Mechanics*, University of Toronto Press, Toronto, 1962, p. 114.

[3] Pars, L.A., *A Treatise on Analytical Dynamics*, Wiley, New York, 1965, pp. 34-37, 528-530.

Reply by Authors to S. F. Felszeghy and C. D. Bailey

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MR. Felszeghy is not strictly correct in stating that we "begin with the assumption that Hamilton's principle is a variational principle." More precisely, we state[1] only that "For the restricted but commonly occurring problem of determining the motion of a conservative system, ..." a variational principle (in the usual meaning of that term) does exist; and the entire paper considers only this restricted problem. Of course, we are aware of the nonextremal nature of the variational principle (our Ref. 1 and, indeed, the entire paper); but this known fact has been overlooked in much of the recent literature on the subject, as noted and documented in the paper.

From our point of view, the only requirement for a variational principle is the existence of a single function V from which both external and internal forces, not including forces of constraint, can be derived. If the system is nonholonomic the virtual displacements will be required to satisfy the constraints; and we have a variational principle with subsidiary conditions (p. 85, our Ref. (4)).

The purpose of our paper is clearly stated in the paper and includes the following goals: 1) to publicize the known nonextremal nature of Hamilton's principle in variational form; 2) in the case of vibration of discrete, conservative, linearly elastic systems, to provide a direct and elementary proof of the known fact that Hamilton's principle *is* a true extremum principle over short time periods; and 3) to give a precise characterization of the maximum length of the time period over which the important result of 2) is guaranteed to be valid.

In order to achieve these goals, it was not necessary to reference Hamilton's original papers or to consider the historical question of whether what is today commonly called Hamilton's principle is indeed what Hamilton presented in 1834 and 1835. Also it was unnecessary to discuss generalizations involving forces which are nonlinear functions of displacement or viscous damping terms.

Professor Bailey states that the term $(\partial T/\partial q_i)\delta q_i\,|_{t_0}^{t_1}$ is not zero in real life. There would appear to be come confusion between the real displacements which a system actually experiences and the virtual, or fictitious, displacements. Real displacements satisfy equations of motion; fictitious

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displacements need not. Subject only to geometric constraints (usually) an analyst is free to pick anything he wants for virtual displacements; and with this freedom, it is very easy to make $(\partial T/\partial q_i)\delta q_i$ equal to zero at any time simply by selecting $\delta q_i = 0$ at that time.

Professor Bailey states that "the expression which is always an extremum is the time integral of the virtual work of the system." This statement is not precise. The fact that Professor Bailey's Eq. (2) indicates a certain quantity equal to zero is not the same as saying that the certain quantity is an extremum. In our paper we do not say that Eq. (1) is the expression which is stationary but rather the quantity I defined by Eq. (2) is stationary because $\delta I = 0$.

Hamilton's "Law of Varying Action" as presented in 1834 and 1835 is the basis of what has since developed into the Hamilton-Jacobi theory (p. 88 of our Ref. 4). This is quite different from the application alluded to by Professor Bailey, whose method, as described in his Ref. 4, is simply an incremental, weighted residual (in particular, Galerkin's method) approximate solution to the differential equations of motion, with trial functions which satisfy initial conditions. Professor Bailey seems to prefer an interpretation in terms of the principle of virtual work because there need be no reference to differential equations. While we agree that virtual work provides a powerful starting point for problems in mechanics, it must be recognized that the differential equation formulation is completely equivalent when necessary continuity conditions are satisfied. Finally, it should be noted that Professor Bailey's approximate solution technique recovers the exact answer to his last example because his family of trial functions happens to include the exact solution. Further comment on his procedure will be more appropriate at another time in another place.

There are two errors in our paper which we take this opportunity to correct. In Eq. (28), the Q_i should be preceded with a minus sign. Then in Eq. (29), the $-Q_i$ should be $+2Q_i$.

References

[1] Smith, D.R. and Smith, C.V. Jr., "When is Hamilton's Principle an Extremum Principle?" *AIAA Journal*, Vol. 12, Nov. 1974, pp. 1573-76.

Further Comment on "Mathematical Modeling of Spinning Elastic Bodies for Modal Analysis

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THE equation of flexural motion of a spinning radial beam is derived in Ref. 1 as an illustration of "partial linearization" advocated by Professor Likins and his associates in describing the structural mode of motion of a flexible spacecraft. In commenting on Ref. 1, Dr. Vigneron[2] indicates the same equation may be obtained in an alternative approach. The two approaches have fundamental disagreements, i.e., the Vigneron derivation considers the change in the kinetic energy of rotation as a result of antenna deflection but neglects the extensional strain energy. The Likins derivation takes exactly the opposite point of view. The formal agreement of the resulting equation could only be considered a coincidence and a closer examination is called for.

For clarity, the Vigneron notation shall be followed. Bending shall be considered in one plane only, i.e., $\eta = 0$. The starting point for both derivations is the Hamilton's principle. The spin is held fixed thus excluding any possibility of the "tail-wags-dog" phenomenon. An Euler-Bernoulli beam is considered. Implicitly or explicitly, the squares of the rotation of the beam cross-section are considered important. Therefore, as it is customary in the "large deflection" of beams, the centroidal axis experiences the strain

$$\epsilon_{yy} = \frac{\partial v}{\partial y} + \frac{1}{2}\left[\frac{\partial \xi}{\partial y}\right]^2 = \frac{\partial \zeta}{\partial y} \tag{1}$$

This equation may be written as

$$\frac{\partial \zeta_0}{\partial y} + \frac{\partial \zeta_1}{\partial y} = \frac{\partial v_0}{\partial y} + \frac{\partial v_1}{\partial y} + \frac{1}{2}\left[\frac{\partial \xi}{\partial y}\right]^2 \tag{2}$$

where, following Likins, the subscript 0 denotes a quantity related to the "undeflected spinning state." The basic difference of the two derivations is that Vigneron considers the centroidal axis inextensible when the beam deflects, i.e.,

$$+\frac{\partial \zeta_1}{\partial y} = \frac{\partial v_1}{\partial y} + \frac{1}{2}\left[\frac{\partial \xi}{\partial y}\right]^2 \approx 0$$

while Likins implicitly assumes $\partial v_1/\partial y \sim 0$, thus implying the centroidal axis experiences additional extension but that elements of the beam will not come closer to the spin axis as a result of deflection. Subsequent derivations of Likins and Vigneron are consistent with their respective assumptions.

Although the ultimate resolution of the issue depends on experimental evidence, it should be observed that: 1) The Likins assumption is inconsistent when the axial equation of motion is considered. 2) The Vigneron derivation really does not depend on the suppression of extensional motion. By considering variations in ζ in the energy expressions given in Ref. 2, one would obtain the following set of *linear* equations, a) the equation for the extensional motion. b) the same flexural equation of motion with the addition of a term representing Coriolis coupling from the extensional motion. As a matter of fact, one may assess the accuracy of the Vigneron assumption based on these equations.

The fact that the Likins derivations happens to give the correct equation of flexural motion may be explained as follows: 1) As noted previously, except for a minor Coriolis coupling, the flexural motion is relatively independent of the extensional motion. 2) The Likins derivation may be interpreted in terms of a formulation in a rotating frame of reference. The kinetic energy of rotation may then be replaced by the centrifugal potential energy which is precisely the term Likins retains as "extensional strain energy."

A key element in the Vigneron formulation is the use of the extensional displacement ζ in place of the radial displacement v as a generalized coordinate. The nonlinear term $\frac{1}{2}(\partial\xi/\partial y)^2$ in Eq. (1) is absorbed in ζ so that although the strain energy in terms of ξ and v are non-quadratic, it becomes quadratic in ξ and ζ and therefore leads to linear equations. Physically this means that although the deflection and the radial motion is coupled when "large deflection" is considered, the coupling between deflection and extensional motion is weaker and the choice of these as variables extends the range of linearity. This is, of course, very attractive and

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